Categorical non abelian cohomology, and the Schreier theory of groupoids

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Abstract

By regarding the classical non abelian cohomology of groups from a 2-dimensional categorical viewpoint, we are led to a non abelian cohomology of groupoids which continues to satisfy classification, interpretation and representation theorems generalizing the classical ones. This categorical approach is based on the fact that if groups are regarded as categories, then, on the one hand, crossed modules are 2-groupoids and, cocycles are lax 2-functors and the cocycle conditions are precisely the coherence axioms for lax 2-functors, and, on the other hand group extensions are fibrations of categories. Furthermore, n-simplices in the nerve of a 2-category are lax 2-functors.

1 Introduction

In this paper the authors have taken up the task of working out and writing down what has been in the mind of a few specialists for quite some time. Although this has been a harder job than we initially imagined, in the end the main point of the paper is making available an *approach* that may be used by people working in the difficult and artificially disconnected fields of higher dimensional category theory and higher non abelian cohomology theory.

It is part of the categorical folklore that *"if your groups are not required to be abelian, you may as well suppose they are groupoids"*. One should think that Schreier's solution (see [11]) to the classification problem of abelian extensions of groups —extended by Dedecker [8] to non abelian extensions— is only part of a larger story. In the whole story, of course, group extensions would be replaced by some sort of groupoid extensions, and the non abelian group cohomology containing the "Schreier invariants" of non abelian group extensions, by a more general cohomology of groupoids.

Since a group extension

$$1 \to K \xrightarrow{j} E \xrightarrow{p} G \to 1 \tag{1}$$

is completely determined by the epimorphism p, and since group epimorphisms are the same as fibrations whose domain and codomain are one-object groupoids, it seems that a categorification of the theory of Schreier invariants of group extensions would begin with the classification of fibrations of groupoids.

One such classification is contained in Grothendieck's work since the 2-category, $\mathsf{Fib}(\mathcal{G})$, of (op)fibrations over any small groupoid $\mathcal{G}$ is 2-equivalent

to the 2-category $\mathsf{LaxFun}(\mathcal{G}, \mathsf{Gpd})$ of lax 2-functors from $\mathcal{G}$ to the 2-category Gpd of groupoids. In one direction, the 2-equivalence is given by the classical Grothendieck construction [9] (called here *twisted product*). An inverse to the Grothendieck construction is obtained (using the axiom of choice) by associating to any given (op)fibration $\mathcal{E} \to \mathcal{G}$ a lax 2-functor "fiber".

The fact that in the study of group extensions such as (1) not only the quotient G but also the kernel K are fixed, indicates that we must classify not all fibrations above a groupoid $\mathcal{G}$ but only those with fixed fibers. On the other hand, the fact that all group morphisms, when they are viewed as functors, are bijective on objects make us reduce our classification of fibrations above a groupoid $\mathcal{G}$ to those which are the identity on objects. This is equivalent to classifying fibrations above $\mathcal{G}$ whose fibers are groups. Therefore, "fixing the kernel" consists in fixing a family of groups $\mathcal{K}$, indexed by the objects of $\mathcal{G}$, and the problem of classifying all fibrations above $\mathcal{G}$ whose fibers are given by the family of groups $\mathcal{K}$ translates, by the above 2-equivalence, to classifying not all lax 2-functors from $\mathcal{G}$ to Gpd but only those which have a fixed image. We observe that the family $\mathcal{K}$ defines a 2-groupoid $\mathsf{AUT}(\mathcal{K})$ (i.e. a crossed module), which is a 2-subcategory of Gpd and that fibrations above $\mathcal{G}$ with fibers $\mathcal{K}$ correspond to lax 2-functors which factor through $\mathsf{AUT}(\mathcal{K})$. Then, we introduce the category $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$ with objects those lax 2-functors from $\mathcal{G}$ to Gpd which factor through $\mathsf{AUT}(\mathcal{K})$ and define a non abelian cohomology of groupoids (which classifies extensions of groupoids by groups) in terms of connected components of lax 2-functors

$$H^2_{|\mathsf{AUT}(\mathcal{K})|}(\mathcal{G}, \mathcal{K}) = H^1\big(\mathcal{G}, \mathsf{AUT}(\mathcal{K})\big) := \widetilde{\mathsf{Act}}[\mathcal{G}, \mathsf{AUT}(\mathcal{K})]$$

so that cocycles will arise from a parameterization of lax 2-functors. In this approach Grothendieck's construction becomes an interpretation theorem of this cohomology in terms of groupoid extensions (Corollary 3.4). Furthermore, the *geometric nerve* of 2-categories can be used to give a representation theorem of this cohomology in terms of homotopy classes of simplicial maps (Theorem 4.2).

Continuing work within this categorical approach is expected to lead to a non abelian 2-dimensional cohomology of 2-groupoids (very closely related to a non abelian 3-dimensional cohomology of groups or groupoids) which would classify extensions of 2-groupoids and which would be represented by homotopy classes of continuous maps from a 2-type to a 3-type.

2 Weak actions of groupoids and twisted products

This is the main section of the paper. The relevant facts of 2-dimensional category theory are presented here with a focus on the concepts (weak actions of groupoids and twisted products) on which the main results are based. By giving a detailed account of weak actions and fibrations we are able to make almost all results in the two subsequent sections to appear either evident or as immediate consequence of the work done here. Although this has made this section to have a comparatively large size we think the effort has been worthwhile.

2.1 2-Categories, lax 2-functors, and lax 2-natural transformations

A 2-*category* $\mathbf{A}$ is just a category enriched in the cartesian closed category Cat of (small) categories. Thus, $\mathbf{A}$ consists of 0-cells (or objects), 1-cells (or arrows) and 2-cells, so that for any two objects $X, Y \in \mathbf{A}$ there is a (small) "hom" category $\mathbf{A}(X, Y)$ whose objects are 1-cells $f : X \to Y$ and whose arrows are 2-cells $\alpha : f \to g$. The composition in all these categories is globally called the *vertical* composition of $\mathbf{A}$, denoted by "$\circ$" or, if there is no ambiguity, by simple juxtaposition. The identity map of f in $\mathbf{A}(X, Y)$ is denoted id_f. Connecting the different hom categories there is a "horizontal composition" *functor*

$$\mathbf{A}(X, Y) \times \mathbf{A}(Y, Z) \xrightarrow{\ *\ } \mathbf{A}(X, Z), \tag{2}$$

satisfying strict associativity, and in each category $\mathbf{A}(X, X)$ there is a distinguished object (i.e. arrow of $\mathbf{A}$), denoted 1_X, which is a strict right and left identity for the horizontal composition. Also, for any three objects X, Y, Z, if f is any given 1-cell $f : X \to Y$, and α is a 2-cell in $\mathbf{A}(Y, Z)$, we use the customary notation $\alpha * f$ or simply αf to denote $\alpha * \mathrm{id}_f$ (and similarly on the other side).

The *underlying category* of $\mathbf{A}$ will be denoted $|\mathbf{A}|$. It has as objects those of $\mathbf{A}$, as arrows the 1-cells of $\mathbf{A}$ and its composition is the horizontal composition of 1-cells in $\mathbf{A}$.

An example of a 2-category is provided by Cat itself. Its 2-cells are the natural transformations, whose vertical composition is essentially given by the composition in the codomain category.

We will make no notational distinction between Cat regarded as a category and Cat regarded as a 2-category since the context will always make it clear in what way it is being considered.

Another example of 2-category is the full 2-subcategory of Cat, denoted Gpd, determined by all small groupoids (categories in which every arrow has an inverse). Also, groups can be regarded as groupoids having only one object (which will be generically denoted "$*$", should it be necessary to refer to it explicitly), and having the group elements as 1-cells. On the other hand, group homomorphisms are precisely the functors between groups regarded as groupoids, so that the category Gp of groups can be regarded as a full subcategory of Gpd. As we do with Cat and with Gpd, we will not make any notational distinction between the category Gp and the full 2-subcategory of Gpd determined by all groups (of which Gp is the underlying category). When Gp is regarded as a 2-subcategory of Gpd, one is implicitly considering as 2-cells $\alpha : f \to g$ between two group homomorphisms $f, g : G \to H$ those elements $\alpha \in H$ representing a natural transformation from f to g, that is, such that for every $u \in G$, the following *naturality condition* holds: $g(u)\,\alpha = \alpha\,f(u)$. Since for any group homomorphism $f : G \to H$ and any element $\alpha \in H$, the map g defined by

$$g(u) = \alpha\,f(u)\,\alpha^{-1} \tag{3}$$

is again a group homomorphism, every such pair (f, α) determines a 2-cell in Gp whose domain is f. Its codomain is the homomorphism $g = \alpha f \alpha^{-1}$ defined by (3). Thus, for any two groups G, H, the 2-cells in $\mathsf{Gp}(G, H)$ are the pairs (f, α)

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where $f : G \to H$ is a group homomorphism and α is an element of H. The vertical composition is essentially given by the group product in the codomain:

$$G \underset{h}{\overset{f}{\Longrightarrow}} H \qquad (g, \beta)(f, \alpha) = (f, \beta\alpha), \tag{4}$$

while the horizontal composition is a sort of semidirect product,

$$G \overset{f}{\underset{h}{\Downarrow \alpha}} H \overset{g}{\underset{k}{\Downarrow \beta}} K \qquad (g, \beta) * (f, \alpha) = \big(gf, \beta g(\alpha)\big). \tag{5}$$

When Gp is regarded as a 2-category, any group G determines a full 2-subcategory of Gp which will be denoted $\mathsf{END}(G)$. Its underlying category is $|\mathsf{END}(G)| = \mathsf{End}(G)$, the category (monoid) of endomorphisms of G.

A 2-*groupoid* $\mathbf{A}$ is a 2-category whose underlying category as well as each of its hom categories are groupoids. This implies that 2-cells are invertible, not only for the vertical composition (for which the inverses are denoted α^{-1}), but also for the horizontal composition, the horizontal inverse of a 2-cell being given by the formulas

$$\mathrm{inv}(\mathrm{id}_f) = \mathrm{id}_{f^{-1}}, \qquad \mathrm{inv}(\alpha) = f^{-1} * \alpha^{-1} * g^{-1} = g^{-1} * \alpha^{-1} * f^{-1}, \tag{6}$$

(where $\alpha : f \to g$ is a 2-cell).

In the category Gp of groups not every 1-cell (group homomorphism) has an inverse. However, all hom categories $\mathsf{Gp}(G, H)$ are groupoids since every 2-cell (f, α) in Gp has an inverse given by

$$(f, \alpha)^{-1} = \big(\alpha f \alpha^{-1}, \alpha^{-1}\big).$$

Thus, if we restrict the 1-cells in Gp to the group isomorphisms, we obtain a sort of (large) 2-groupoid which will be denoted $\mathsf{Iso}(\mathsf{Gp})$.

In the same way that a group G determines a full 2-subcategory of Gp, it also determines a full 2-subcategory of $\mathsf{Iso}(\mathsf{Gp})$, denoted $\mathsf{AUT}(G)$, whose underlying category is $|\mathsf{AUT}(G)| = \mathsf{Aut}(G)$, the group of automorphisms of G. Note that $\mathsf{AUT}(G)$ is automatically a 2-groupoid. We will find it useful to generalize the above notation to arbitrary families of groups, so that if $\mathcal{K} = \{K_x\}_{x \in X}$ is a family of groups indexed by a set X, the full 2-subcategory of $\mathsf{Iso}(\mathsf{Gp})$ determined by $\mathcal{K}$ will be denoted $\mathsf{AUT}(\mathcal{K})$. As in the previous case, this is actually a 2-groupoid.

A 2-*functor* $F : \mathbf{A} \to \mathbf{B}$ is an enriched functor in Cat, so it takes objects, 1-cells and 2-cells in $\mathbf{A}$ to objects, 1-cells and 2-cells in $\mathbf{B}$ respectively, in such a way that all the 2-category structure of $\mathbf{A}$ is strictly preserved. In particular, any 2-functor preserves inverses of 1-cells and of 2-cells.

Small 2-categories and 2-functors form a category that we denote 2-Cat (this is actually a 2-category and even a 3-category, but we will not use in this paper those higher dimensional structures). The full subcategory of 2-Cat determined by the 2-groupoids is denoted by 2-Gpd.

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Example 2.1 (2-functors $G \to \mathsf{AUT}(K)$)
Given any two groups G and K we can consider 2-functors $G \to \mathsf{AUT}(K)$. Since G is 2-discrete as a 2-groupoid, to give one such 2-functor is the same as giving a group homomorphism $G \to \mathsf{Aut}(K)$, that is, a group action of G on K.

For any small category $\mathcal{C}$, the category of all actions of $\mathcal{C}$ on groups is defined as the functor category $\mathsf{Gp}^{\mathcal{C}}$. If $\mathcal{C}$ has only one object an action of $\mathcal{C}$ on groups (regarded as a functor $F : \mathcal{C} \to \mathsf{Gp}$) determines the specific group on which $\mathcal{C}$ acts as the image $F(*)$ of the only object of $\mathcal{C}$. This allows us to parameterize the actions $\mathcal{C}$ by the groups acted upon. This is one of the ways in which one can arrive at the concept of an action of $\mathcal{C}$ on a specific group H. If we drop all restriction on the number of objects of $\mathcal{C}$, an action of $\mathcal{C}$ on groups determines, in general, not a group, but *a family of groups indexed by the objects of $\mathcal{C}$*. This allows us to parameterize the actions of $\mathcal{C}$ on groups by the $\mathcal{C}_{\mathrm{obj}}$-indexed families of groups on which $\mathcal{C}$ may act. (Throughout the paper we will use the notation $\mathcal{C}_{\mathrm{obj}}$ to denote the set of objects of a small category $\mathcal{C}$.)

Definition 2.2
If $\mathcal{G}$ is a groupoid and $\mathcal{K}$ is a family of groups indexed by the objects of $\mathcal{G}$, a groupoid action of $\mathcal{G}$ on $\mathcal{K}$ is a 2-functor $\mathcal{G} \to \mathsf{AUT}(\mathcal{K})$ (or, equivalently, a functor $\mathcal{G} \to |\mathsf{AUT}(\mathcal{K})|$) whose objects function is the indexing of $\mathcal{K}$, that is, such that for every object $A \in \mathcal{G}$, $K_A = F(A)$.

We come now to the maps between 2-categories that are crucial in this paper. These are the *lax* 2-functors, whose definition we recall next:

Definition 2.3
Given two 2-categories $\mathbf{A}$, $\mathbf{B}$, a (normal[1]) lax 2-functor from $\mathbf{A}$ to $\mathbf{B}$ is a pair (F, σ) where

(a) *F is a correspondence which takes each object X of $\mathbf{A}$ to an object $F(X)$ of $\mathbf{B}$, and any two objects X, Y in $\mathbf{A}$, to a functor $F_{XY} : \mathbf{A}(X, Y) \to \mathbf{B}\big(F(X), F(Y)\big)$.*
(b) *σ is a correspondence which assigns to any three objects X, Y, Z in $\mathbf{A}$ a natural transformation σ^{XYZ} between functors from $\mathbf{A}(X, Y) \times \mathbf{A}(Y, Z)$ to $\mathbf{B}\big(F(X), F(Z)\big)$, whose component on an "object" $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a 2-cell in $\mathbf{B}$ denoted $\sigma^{XYZ}_{gf} : F_{XZ}(gf) \to F_{YZ}(g)F_{XY}(f)$.*

We will generally omit the subscripts in F_{XY} and the superscripts in σ^{XYZ}, the objects being implicit, so that the above component of σ will be written

$$\sigma_{gf} : F(gf) \to F(g)F(f). \tag{7}$$

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[1]Note that in this paper all lax 2-functors will be normalized. Thus, the expression "lax 2-functor" will mean here what is usually referred to in the literature as "normal lax 2-functor", that is, for us a lax 2-functor is strictly identity preserving. The structural natural transformation for identities is omitted from the definition since it is itself an identity. In exchange, we need to include an additional requirement of "coherence with the identity law" for the structural natural transformation for composition, σ.

The naturality of σ gives, for each pair of horizontally composable 2-cells

$$(\alpha, \beta) : (f, g) \to (h, k), \quad \text{or} \quad \cdot \underset{h}{\overset{f}{\Downarrow \alpha}} \cdot \underset{k}{\overset{g}{\Downarrow \beta}} \cdot ,$$

the following equation:

$$\sigma_{kh} \, F(\beta * \alpha) = \big(F(\beta) * F(\alpha)\big) \, \sigma_{gf}. \tag{8}$$

$$
\begin{array}{ccc}
(f, g) & F(gf) \xrightarrow{\ \sigma_{gf}\ } F(g)F(f) \\
(\alpha,\beta) \Big\downarrow & F(\beta*\alpha) \Big\downarrow \qquad\qquad \Big\downarrow F(\beta)*F(\alpha) \ . \\
(h, k) & F(kh) \xrightarrow[\ \sigma_{kh}\ ]{} F(k)F(h)
\end{array}
$$

These data are required to satisfy the following axioms:

LF1. *(Normalization)* $F(1_X) = 1_{F(X)}$.

LF2. *(Coherence with associative law) For any 3-chain in $\mathbf{A}$, $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$,*

$$\big(\sigma_{hg} * F(f)\big) \, \sigma_{(hg)f} = \big(F(h) * \sigma_{gf}\big) \, \sigma_{h(gf)}, \tag{9}$$

$$
\begin{array}{ccc}
F(hgf) & \xrightarrow{\ \sigma_{(hg)f}\ } & F(hg)F(f) \\
\sigma_{h(gf)} \Big\downarrow & & \Big\downarrow \sigma_{hg}*F(f) \\
F(h)F(gf) & \xrightarrow[\ F(h)*\sigma_{gf}\ ]{} & F(h)F(g)F(f).
\end{array}
$$

LF3. *(Coherence with identity law) For any 1-cell $f : X \to Y$,*

$$\sigma_{f1_X} = \mathrm{id}_{F(f)} = \sigma_{1_Y f}.$$

Small 2-categories and lax functors form a category that we denote by 2-$\mathsf{Cat}_{\mathrm{lax}}$, analogously 2-$\mathsf{Gpd}_{\mathrm{lax}}$ will be the full subcategory of 2-$\mathsf{Cat}_{\mathrm{lax}}$ whose objects are the 2-groupoids.

Proposition 2.4
In the presence of inverses of 2-cells in the codomain 2-category, axiom LF3 in Definition 2.3 can be substituted by the following weaker version:

LF3'. *(Weak coherence with identity law) For any object X,*

$$\sigma_{1_X 1_X} = \mathrm{id}_{F(1_X)} (= \mathrm{id}_{1_{F(X)}}).$$

Proof: Putting $g = h = 1_Y$ in (9) and using axiom LF1 and the inverse 2-cell of $\sigma_{1_Y f}$, one gets $\sigma_{1_Y f} = \sigma_{1_Y 1_Y} * F(f) = \sigma_{1_Y 1_Y} * \mathrm{id}_{F(f)}$. Therefore axiom LF3' implies $\sigma_{1_Y f} = \mathrm{id}_{F(f)}$. Similarly, putting $f = g = 1_Z$ in (9) and using axiom LF1 and the inverse 2-cell of σ_{h1_Z}, one gets $\sigma_{h1_Z} * F(1_Z) = F(h) * \sigma_{1_Z 1_Z} = \mathrm{id}_{F(h)} * \sigma_{1_Z 1_Z}$. Therefore axiom LF3' implies $\sigma_{h1_Z} = \mathrm{id}_{F(h)}$. ∎

Since we have enriched the group $\mathsf{Aut}(K)$ of automorphism of a group K to the 2-category $\mathsf{AUT}(K)$, in such a way that a group action of a group G on K is just a 2-functor $G \to \mathsf{AUT}(K)$, we can relax the definition of action by considering lax 2-functor $G \to \mathsf{AUT}(K)$ which are just weak actions of G on K. Moreover there is no difficulty to passing from group to groupoids.

Definition 2.5 (Weak groupoid actions)

If $\mathcal{G}$ is a groupoid and $\mathcal{K} = \{K_A\}_{A \in \mathcal{G}}$ is a family of groups indexed by the objects of $\mathcal{G}$, a weak action of $\mathcal{G}$ on $\mathcal{K}$ is a lax 2-functor $F : \mathcal{G} \to \mathsf{AUT}(\mathcal{K})$ whose objects function is the indexing of $\mathcal{K}$, that is, such that for every object $A \in \mathcal{G}$, $K_A = F(A)$.

If G is a group, the condition on the lax 2-functor $G \to \mathsf{AUT}(K)$ to be a lax action is always satisfied and therefore a lax group action of G on K is just a lax 2-functor $G \to \mathsf{AUT}(K)$.

The following proposition spells out the data and the axioms characterizing weak actions of groupoids.

Proposition 2.6

Given a groupoid $\mathcal{G}$ and a family of groups $\mathcal{K}$ indexed by the objects of $\mathcal{G}$, a weak action of $\mathcal{G}$ on $\mathcal{K}$ is a pair $\langle F, \sigma \rangle$ where F is a correspondence assigning to each arrow $u : A \to B$ in $\mathcal{G}$ a group isomorphism $F(u) = {}^u(-) : K_A \to K_B$, and σ is a correspondence assigning to each composable pair $A \xrightarrow{u} B \xrightarrow{v} C$ in $\mathcal{G}$ a group element $\sigma_{vu} \in K_C$ such that, for each 3-chain $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} D$ in $\mathcal{G}$, and any element $x \in K_A$, the following equations are satisfied,

1. ${}^1x = x$,
2. $\sigma_{11} = 1$,
3. $\sigma_{vu}{}^{vu}x = {}^v({}^ux)\sigma_{vu}$,
4. $\sigma_{wv}\sigma_{(wv)u} = {}^w\sigma_{vu}\sigma_{w(vu)}$.

Proof: Condition 3 can be rewritten $F(v) \circ F(u) = \sigma_{vu}F(vu)\sigma_{vu}^{-1}$ and it assures, according to equation (3) (page 3), that σ_{vu} is a 2-cell from $F(vu)$ to $F(v) \circ F(u)$. Since $\mathcal{G}$ is 2-discrete, the naturality condition (8) is vacuous and therefore σ is a natural transformation. Condition 1 is the normalization of F, condition 2, the coherence with the identity law, and condition 4, the coherence with the associative law. ∎

If G is a group, a weak action of G on a group H consists of a pair of maps:

$$F : G \to \mathsf{Aut}(K), \qquad \sigma : G \times G \to K$$

satisfying the conditions in Proposition 2.6 (in this context $\sigma(u, v)$ will denote the element σ_{vu}). This is just a Dedecker's 2-cocycle of G with coefficients on the crossed module $K \to \mathsf{Aut}(K)$ associated to the 2-groupoid $\mathsf{AUT}(K)$.

The natural morphisms between lax 2-functors are the lax 2-natural transformations, whose definition we recall next.

Definition 2.7

Given lax 2-functors $F_1, F_2 : \mathbf{A} \to \mathbf{B}$, a lax 2-natural transformation from F_1 to F_2 is a pair $\boldsymbol{\alpha} = (\alpha, \tau^\alpha)$ where

(a) α is a correspondence which assigns to each object $A \in \mathbf{A}$ an arrow $\alpha_A : F_1(A) \to F_2(A)$ in $\mathbf{B}$, and

(b) τ^α is a correspondence which assigns to each two objects A, B in $\mathbf{A}$ a natural transformation between functors from $\mathbf{A}(A, B)$ to $\mathbf{B}\big(F_1(A), F_2(B)\big)$, whose component at a morphism $f : A \to B$ in $\mathbf{A}$ is a 2-cell, τ^α_f, in $\mathbf{B}$ as in the diagram

$$
\begin{array}{ccc}
F_1(A) & \xrightarrow{\alpha_A} & F_2(A) \\
{\scriptstyle F_1(f)}\downarrow & \overset{\tau^\alpha_f}{\Longrightarrow} & \downarrow{\scriptstyle F_2(f)} \\
F_1(B) & \xrightarrow[\alpha_B]{} & F_2(B)
\end{array}
\qquad \tau^\alpha_f : \alpha_B\, F_1(f) \to F_2(f)\, \alpha_A,
$$

the naturality of τ^α gives, for each arrow $\eta : f \to g$ in $\mathbf{A}(A, B)$ the following equation:

$$
\big(F_2(\eta) * \alpha_A\big)\, \tau^\alpha_f = \tau^\alpha_g \big(\alpha_B * F_1(\eta)\big).
$$

These data are to satisfy the axioms:

1. (normalization) $\tau^\alpha_{1_A} = \mathrm{id}_{\alpha_A}$,

2. (coherence) for each composable pair of arrows $A \xrightarrow{f} B \xrightarrow{g} C$,

$$
\big(F_2(g) * \tau^\alpha_f\big)\big(\tau^\alpha_g * F_1(f)\big)\big(\alpha_C * \sigma_{1gf}\big) = \big(\sigma_{2gf} * \alpha_A\big)\tau^\alpha_{(gf)}.
$$

An example of lax 2-natural transformation is given by the *identity lax 2-natural transformation* of a lax 2-functor $F : \mathbf{A} \to \mathbf{B}$, which is defined by $\alpha_A = 1_{F(A)}$ and $\tau_f = \mathrm{id}_{F(f)}$.

Given two lax 2-natural transformations $F_1 \xrightarrow{\boldsymbol{\alpha}} F_2 \xrightarrow{\boldsymbol{\beta}} F_3$ between lax 2-functors $\mathbf{A} \to \mathbf{B}$, a composite lax 2-natural transformation, $\boldsymbol{\beta\alpha} : F_1 \to F_3$, can be defined with structure given by

$$
\tau^{\beta\alpha}_f = (\tau^\beta_f * \alpha_A)(\beta_B * \tau^\alpha_f).
$$

This composition is associative and it has the identity lax 2-natural transformations as two-sided identities so that for any two 2-categories $\mathbf{A}$, $\mathbf{B}$ there is a category, denoted $\mathsf{LaxFun}(\mathbf{A}, \mathbf{B})$, whose objects are the lax 2-functors $\mathbf{A} \to \mathbf{B}$ and whose arrows are the lax 2-natural transformations between them. By taking lax 2-natural transformations as 2-cells, the category $2\text{-}\mathsf{Cat}_{\mathrm{lax}}$ of small 2-categories and lax 2-functors becomes a 2-category.

Example 2.8 (Maps in $\mathsf{LaxFun}\big(\mathcal{G}, \mathsf{AUT}(\mathcal{K})\big)$)
Let $\mathcal{G}$ be a groupoid, $\mathcal{K} = \{K_A\}_{A\in\mathcal{G}}$ a family of groups indexed by the objects of $\mathcal{G}$, and let $(F_1, \sigma^1), (F_2, \sigma^2) : \mathcal{G} \to \mathsf{Gp}$ be lax 2-functors from $\mathcal{G}$ to $\mathsf{AUT}(\mathcal{K})$. A lax 2-natural transformation $\boldsymbol{\alpha} : F_1 \to F_2$ is a pair $\boldsymbol{\alpha} = (\alpha, \tau)$ where for every object $A \in \mathcal{G}$, $\alpha_A : K_A \to K_A$ is a group isomorphism, and for every pair of objects $A, B \in \mathcal{G}$, $\tau_{AB} : \mathcal{G}(A, B) \to K_B$ is a map of sets satisfying for all $k \in K_x$, and $u \in \mathcal{G}(A, B)$,

1. (naturality) $\quad F_2(u)\big(\alpha_A(k)\big) \cdot \tau_{AB}(u) = \tau_{AB}(u) \cdot \alpha_B\big(F_1(u)(k)\big)$,
2. (normalization) $\tau_{AA}(1) = 1$, and

3. *(coherence)* $\quad F_2(v)\big(\tau_{AB}(u)\big) \cdot \tau_{BC}(v) \cdot \alpha_C(\sigma^1_{vu}) = \sigma^2_{vu} \cdot \tau_{AC}(vu)$

We next define the category of weak actions of a groupoid $\mathcal{G}$ on a family $\mathcal{K}$ of groups indexed by the objects of $\mathcal{G}$.

Definition 2.9

Let $\mathcal{G}$ be a groupoid, and $\mathcal{K} = \{K_A\}_{A \in \mathcal{G}}$ a $\mathcal{G}_{obj}$-indexed family of groups. If F_1, F_2 are two weak actions of $\mathcal{G}$ on $\mathcal{K}$, a map of weak actions $F_1 \to F_2$ in the category $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$ is a lax 2-natural transformations $(\alpha, \tau) : F_1 \to F_2$ such that for every object $A \in \mathcal{G}$, $\alpha_A = 1_{K_A}$.

According to this definition, there is an inclusion of categories

$$\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K}) \subseteq \mathsf{LaxFun}\big(\mathcal{G}, \mathsf{AUT}(\mathcal{K})\big)$$

which is an identity on objects. In the next example we see how the axioms of lax 2-natural transformations get simplified in the case of weak actions of a group.

Example 2.10 (Morphisms of weak actions of a group)

Let G and K be groups and let $(F_1, \sigma_1), (F_2, \sigma_2) : G \to \mathsf{AUT}(K)$ be weak actions of G on K. To give a morphism $F_1 \to F_2$ in $\widetilde{\mathsf{Act}}(G, K)$ is to give a map of sets $\tau : G \to K$ satisfying for all $k \in K$, and $u, v \in G$,

1. *(naturality)* $\quad \tau(u) \cdot F_1(u)(k) = F_2(u)(k) \cdot \tau(u)$,
2. *(normalization)* $\tau(1) = 1$,
3. *(coherence)* $\quad F_2(v)\big(\tau(u)\big) \cdot \tau(v) \cdot \sigma_1(u, v) = \sigma_2(u, v) \cdot \tau(vu)$

Note that by the naturality condition, F_2 is completely determined by τ and F_1 and that the coherence condition determines σ_2:

$$F_2(u)(k) = \tau(u) \cdot F_1(u)(k) \cdot \tau(u)^{-1} \tag{10}$$

$$\sigma_2(u, v) = \tau(v) \cdot F_1(v)\big(\tau(u)\big) \cdot \sigma_1(u, v) \cdot \tau(vu)^{-1}. \tag{11}$$

If we write $C^1(G, K)$ for the set of maps $\tau : G \to K$ such that $\tau(1) = 1$ and $C^2_{\mathsf{Aut}(K)}(G)$ for the set of arrows in $\widetilde{\mathsf{Act}}(G, K)$ we can define an operation

$$\nabla : C^1(G, K) \times C^2_{\mathsf{Aut}(K)}(G) \longrightarrow C^2_{\mathsf{Aut}(K)}(G)$$

as $\nabla(\tau, F_1) = \tau \nabla F_1 = (F_2, \sigma_2)$ where F_2 y σ_2 are defined by equations (10) and (11). The map ∇ is one of the three operations defined by Dedecker in [8] where it is used to define cohomologous cocycles.

Example 2.11 (Morphisms of weak actions of a groupoid)

Let $\mathcal{G}$ be a groupoid, let $\mathcal{K} = \{K_A\}_{A \in \mathcal{G}}$ be a family of groups indexed by the objects of $\mathcal{G}$, and let $(F_1, \sigma^1), (F_2, \sigma^2) : \mathcal{G} \to \mathsf{Gp}$ be weak actions of $\mathcal{G}$ on $\mathcal{K}$. To give a morphism $F_1 \to F_2$ in $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$ is to give for every pair of objects $A, B \in \mathcal{G}$, a map of sets $\tau_{AB} : \mathcal{G}(A, B) \to K_B$ satisfying, for all $k \in K_A$, and $A \xrightarrow{u} B \xrightarrow{v} C$ in $\mathcal{G}$,

1. *(naturality)* $\quad \tau_{AB}(u) \cdot F_1(u)(k) = F_2(u)(k) \cdot \tau_{AB}(u)$,

2. *(normalization)* $\tau_{AA}(1) = 1$,

3. *(coherence)* $\quad F_2(v)\big(\tau_{AB}(u)\big) \cdot \tau_{BC}(v) \cdot \sigma^1_{vu} = \sigma^2_{vu} \cdot \tau_{AC}(vu)$

As in the group case, the codomain of a morphism of weak actions of $\mathcal{G}$ on $\mathcal{K}$ is completely determined by its source and the family of maps τ_{AB}. Then, if we write $C^1(\mathcal{G}, \mathcal{K})$ for the set of families of maps $\boldsymbol{\tau} = \{\tau_{AB}\}_{A,B \in \mathcal{G}}$ such that $\tau_{AA}(1) = 1$ for all object $A \in \mathcal{G}$, and $C^2_{\mathsf{AUT}(\mathcal{K})}(\mathcal{G})$ for the set of arrow in $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$, we can define an operation

$$\nabla : C^1(\mathcal{G}, \mathcal{K}) \times C^2_{\mathsf{AUT}(\mathcal{K})}(\mathcal{G}) \longrightarrow C^2_{\mathsf{AUT}(\mathcal{K})}(\mathcal{G})$$

as $\nabla(\boldsymbol{\tau}, F_1) = \boldsymbol{\tau} \nabla F_1 = (F_2, \sigma^2)$ where F_2 y σ^2 are obtained by solving for them in the naturality and coherence conditions above.

2.2 Fibrations and the lax 2-functor "fiber" associated to a fibration

Given a functor, $F : \mathcal{A} \to \mathcal{G}$, the fiber of F over an object $B \in \mathcal{G}$ is the subcategory $F^*(B)$ of $\mathcal{A}$ defined by the following pullback in Cat:

$$\begin{array}{ccc} F^*(B) & \longrightarrow & \mathbf{1} \\ \downarrow & {\scriptstyle \text{p.b.}} & \downarrow {\scriptstyle B} \\ \mathcal{A} & \xrightarrow{\ F\ } & \mathcal{G} \end{array} \qquad (12)$$

Thus, $F^*(B)$ has:

- objects: those objects $A \in \mathcal{A}$ such that $F(A) = B$, and
- arrows: those arrows $f \in \mathcal{A}$ such that $F(f) = \mathrm{id}_B$.

A fibration is a functor for which the process of "taking fibers" is as functorial as it can be.

Definition 2.12
A functor between groupoids $F : \mathcal{A} \to \mathcal{G}$ is called a (Grothendieck op-) fibration if for any object $A \in \mathcal{A}$ and any arrow $f : F(A) \to B$ in $\mathcal{G}$ there exists an arrow $\widetilde{f} : A \to \widetilde{A}$ in $\mathcal{A}$ such that $F(\widetilde{f}) = f$.

We will write $\mathsf{Fib}(\mathcal{G})$ for the full subcategory of the slice category $\mathsf{Gpd}/\mathcal{G}$ determined by those objects which are fibrations.

Example 2.13 (The lax 2-functor "fiber" of a fibration of groupoids)
Given a fibration $F : \mathcal{A} \to \mathcal{G}$ of groupoids, let us suppose we have chosen for each pair (f, A) $\big($where $f : X \to Y$ is an arrow of $\mathcal{G}$ and A is an object of $F^*(X)\big)$, an arrow $\widetilde{f}_A$ with domain A and such that $F(\widetilde{f}_A) = f$ (if f is an identity, the choice is $(\widetilde{1_X})_A = 1_A$). Let's denote by ${}^f A$ the codomain of $\widetilde{f}_A$ so that we have $\widetilde{f}_A : A \to {}^f A$, and

$$\begin{array}{ccc} A & & X \\ {\scriptstyle \widetilde{f}_A}\big\downarrow & \overset{F}{\mapsto} & \big\downarrow{\scriptstyle f} \\ {}^f A & & Y \end{array}$$

Based on the above choices one can give a lax 2-functor "fiber of F" (which is not unique, since it depends on a choice),

$$(F^*, \sigma) : \mathcal{G} \to \mathsf{Gpd},$$

defined in the following way:

F^*: *On objects, via the pullback (12); on the arrows $f : X \to Y$ in $\mathcal{G}$ as the functor $F^*(f) : F^*(X) \to F^*(Y)$ defined by (where $h : A \to B$ in $F^*(X)$)*

$$F^*(f)(A) = {}^f A, \qquad F^*(f)(h) = \widetilde{f}_B h \widetilde{f}_A^{-1}. \tag{13}$$

σ: *Its component on a composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{G}$ is the natural transformation $\sigma_{gf} : F^*(gf) \to F^*(g)F^*(f)$ whose component at an object A of $F^*(X)$, is the arrow $\sigma_{gf}^A : {}^{gf}A \to {}^g({}^f A)$, defined by*

$$\sigma_{gf}^A = \widetilde{g}_{({}^f A)} \, \widetilde{f}_A \, \widetilde{(gf)}_A^{-1}.$$

It is a simple exercise to prove that equations (13) define a functor $F^*(f)$, that σ_{gf} is indeed a natural transformation, and that σ satisfies the conditions for the structure map of a lax 2-functor (note that in this context one gets the equation

$$\sigma_{hg}^{{}^f A} \, \sigma_{(hg)f}^A = F^*(h)\big(\sigma_{gf}^A\big) \, \sigma_{h(gf)}^A, \tag{14}$$

for the associativity coherence of σ), so that the above data (F^*, σ) indeed determines a lax 2-functor.

Observation 2.14
It is noteworthy the fact that for each arrow $f : X \to Y$ in $\mathcal{G}$, the functor $F^*(f)$ is bijective at the level of arrows. That is, for any two objects $A, B \in F^*(X)$ the map

$$F^*(f) : F^*(X)(A, B) \longrightarrow F^*(Y)({}^f A, {}^f B)$$

defined by (13) is a bijection (the inverse map sends k to $\widetilde{f}_B^{-1} k \widetilde{f}_A$). In particular, for any object $A \in \mathcal{A}$ and any arrow $f : X \to Y$ in $\mathcal{G}$, $F^*(f)$ determines a group isomorphism between the kernels of the group homomorphisms $F_{AA} : \mathcal{A}(A, A) \to \mathcal{G}(X, X)$ and $F_{{}^f A {}^f A} : \mathcal{A}({}^f A, {}^f A) \to \mathcal{G}(Y, Y)$.

Example 2.15 (Case of a fibration bijective on objects)
If a fibration of groupoids $F : \mathcal{E} \to \mathcal{G}$ is bijective on objects, every fiber has only one object and it is therefore a group, so that any fiber functor F^* of F goes to Gp. Furthermore, by Observation 2.14, every arrow of $\mathcal{G}$ is sent by F^* to a group isomorphism. The data that needs to be given together with F in order to determine a lax functor "fiber" $F^* : \mathcal{G} \to \mathsf{Gpd}$ consists in choosing, for each arrow $f \in \mathcal{G}$, an arrow $\widetilde{f} \in \mathcal{E}$ such that $F(\widetilde{f}) = f$, and $\widetilde{1} = 1$, that is, we need to specify a set-theoretic section of the arrows function of F such that it takes the identities to identities. Once we have made this choice, the lax 2-functor $F^* : \mathcal{G} \to \mathsf{Iso}(\mathsf{Gp})$ takes an object A of $\mathcal{G}$ to the kernel K_A of the group homomorphism $F_{AA} : \mathcal{E}(A, A) \to \mathcal{G}(A, A)$,[2] each arrow $f : A \to B$ in $\mathcal{G}$ to the group isomorphism

$$F^*(f) = {}^f(-) : K_A \to K_B; \quad {}^f k = \widetilde{f} k \widetilde{f}^{-1},$$

---

[2]It is harmless to assume that the (bijective) object function of F is an identity.

and the structure map of F^* is given by

$$\sigma_{gf} = \widetilde{g}\,\widetilde{f}\,\left(\widetilde{gf}\right)^{-1}.$$

Is F is a fibration of groupoids which is bijective on objects, any lax 2-functor fiber F^* determines a *weak action* of the base groupoid on the family of groups determined by the fibers. In the particular case of groups this reduces to the fact that any section of an epimorphism of groups determines a weak action of the codomain on the kernel.

Example 2.16 (The lax 2-functor fiber of a fibration of groups)
A fibration between groups is just a surjective homomorphism $p : E \twoheadrightarrow G$. The fiber of p over the only object of G is obviously the kernel of p. A lax 2-functor "fiber" $p^* : G \to \mathsf{AUT}(K)$ is determined by choosing, for each element $u \neq 1$ in G, an element $\widetilde{u} \in E$ such that $p(\widetilde{u}) = u$, or by specifying a set-theoretic section of p taking the identity of G to the identity of E. Having made this choice, p^* is defined as taking the only object of G to $K = \ker(p)$, each element $u \in G$ to the following automorphism of K:

$$p^*(u) = {}^u(-) : K \to K ; \quad {}^u x = \widetilde{u}\, x\, \widetilde{u}^{-1},$$

and the structure map of p^* is given by

$$\sigma : G \times G \to K ; \qquad \sigma(u,v) = \sigma_{vu} = \widetilde{v}\,\widetilde{u}\,\widetilde{vu}^{-1}.$$

2.3   Twisted Products

We have seen in Example 2.13 how fibrations of groupoids give rise to lax 2-functors to the 2-category Gpd. The inverse process produces a fibration of groupoids from a lax 2-functor

$$F : \mathcal{G} \to \mathsf{Gpd}. \tag{15}$$

Using the terminology of Schreier [11], one can call the resulting fibration the *twisted product* of $\mathcal{G}$ and F, since it reduces to the case dealt with by Schreier in the case that $\mathcal{G}$ is a group, $H = F(*)$ is also a group. The domain of the obtained fibration is (in the case of groups) the group twisted product of $\mathcal{G}$ and H relative to the weak action induced by F. Although we are using Schreier terminology we will use Grothendieck notation, so that we write $\int_{\mathcal{G}} F$ for the domain of the obtained fibration. Let's review this construction in the particular case at hand.

Given the lax 2-functor (15), if $f : X \to Y$ is an arrow in $\mathcal{G}$, the action of the functor $F(f) : F(X) \to F(Y)$ will be denoted by a left action notation, ${}^f(-)$. Accordingly, for any arrow $\lambda : A \to B$ in $F(X)$, we write $F(f)(\lambda) = \left({}^f\lambda : {}^fA \to {}^fB\right)$.

Definition 2.17 (Twisted product)
Given a groupoid $\mathcal{G}$ and a lax 2-functor $(F, \sigma) : \mathcal{G} \to \mathsf{Gpd}$, the twisted product groupoid $\int_{\mathcal{G}} F$ has as objects the pairs (X, A) where X is an object in $\mathcal{G}$ and A is an object in $F(X)$. The arrows of $\int_{\mathcal{G}} F$ are also pairs $(f, \lambda) : (X, A) \to (Y, B)$

where $f : X \to Y$ is an arrow in $\mathcal{G}$ and $\lambda : {}^{f}A = F(f)(A) \to B$ is an arrow in $F(Y)$. The composition of arrows

$$(X, A) \xrightarrow{(f,\lambda)} (T, B) \xrightarrow{(g,\mu)} (Z, C)$$

is defined as

$$(g, \mu)\,(f, \lambda) = \left(gf, \mu\ {}^{g}\lambda\ \sigma^{A}_{gf}\right)$$

$${}^{(gf)}A \xrightarrow{\sigma^{A}_{gf}} {}^{g}({}^{f}A) \xrightarrow{{}^{g}\lambda} {}^{g}B \xrightarrow{\mu} C\ ,$$

while the identity map of (X, A) is $(1_X, 1_A)$. This gives a category $\int_{\mathcal{G}} F$ which is a groupoid since every arrow (f, λ) has an inverse, which is defined by

$$(f, \lambda)^{-1} = \left(f^{-1}, \left({}^{f^{-1}}\lambda\ \sigma^{A}_{f^{-1}f}\right)^{-1}\right).$$

(That this is a *right* inverse to (f, λ) is the non trivial part of the proof. It requires the identity ${}^{f}\left(\sigma^{A}_{f^{-1}f}\right) = \sigma^{A}_{ff^{-1}}$ which is a consequence of (14) with $h = f$ and $g = f^{-1}$.)

The obvious projection $\int_{\mathcal{G}} F \to \mathcal{G}$ is a fibration of groupoids, with fiber over any object $X \in \mathcal{G}$ the groupoid $F(X)$ (after the opportune identification of the objects $A \in F(X)$ with the objects $(X, A) \in \int_{\mathcal{G}} F$ and similarly for the arrows).

The twisted product construction just defined is moreover functorial, that is, one can define, for each lax 2-natural transformation $\boldsymbol{\alpha} : F_1 \to F_2$ in $\mathsf{LaxFun}(\mathcal{G}, \mathsf{Gpd})$ a *functor of fibrations* above $\mathcal{G}$,

$$\int_{\mathcal{G}} \boldsymbol{\alpha} : \int_{\mathcal{G}} F_1 \longrightarrow \int_{\mathcal{G}} F_2$$

so that given $F_1 \xrightarrow{\boldsymbol{\alpha}} F_2 \xrightarrow{\boldsymbol{\beta}} F_3$, $\int_{\mathcal{G}}(\boldsymbol{\beta\alpha}) = \left(\int_{\mathcal{G}} \boldsymbol{\beta}\right)\left(\int_{\mathcal{G}} \boldsymbol{\alpha}\right)$.

So, we get a functor

$$\int_{\mathcal{G}} : \mathsf{LaxFun}(\mathcal{G}, \mathsf{Gpd}) \longrightarrow \mathsf{Fib}(\mathcal{G}) \tag{16}$$

Let $F : \mathcal{E} \to \mathcal{G}$ be a fibration of groupoids. Then, applying the twisted product construction to any fiber lax functor $F^{*} : \mathcal{G} \to \mathsf{Gpd}$ obtained from F produces a new fibration $\int_{\mathcal{G}} F^{*} \to \mathcal{G}$ such that there is a functor $\Gamma : \mathcal{E} \to \int_{\mathcal{G}} F^{*}$ which makes commutative the triangle

$$\mathcal{E} \xrightarrow{\ \Gamma\ } \int_{\mathcal{G}} F^{*}$$
$$F \searrow \quad \swarrow$$
$$\mathcal{G}$$

that is, Γ is an arrow in $\mathsf{Fib}(\mathcal{G})$.

Furthermore Γ is an isomorphism of categories, so that any fibration of groupoids can be recovered (up to isomorphism) as the canonical projection from a "*twisted product*". We have the following important well known result:

Proposition 2.18
The functor $\int_{\mathcal{G}} : \mathsf{LaxFun}(\mathcal{G}, \mathsf{Gpd}) \to \mathsf{Fib}(\mathcal{G})$ is an equivalence of categories.

13

Example 2.19 (Twisted product of weak actions)
For a lax 2-functor $F : \mathcal{G} \to \mathsf{Gp}$, the twisted product $\int_{\mathcal{G}} F$ is a groupoid with the "same objects" as $\mathcal{G}$ and the twisted product fibration $\int_{\mathcal{G}} F \to \mathcal{G}$ is the identity on objects. Conversely, if $F : \mathcal{E} \to \mathcal{G}$ is a fibration of groupoids which is bijective on objects, then any fiber lax 2-functor of F goes to groups. Therefore, the equivalence of categories

$$\int_{\mathcal{G}} : \mathsf{LaxFun}(\mathcal{G}, \mathsf{Gpd}) \longrightarrow \mathsf{Fib}(\mathcal{G})$$

restricts to an equivalence of categories

$$\int_{\mathcal{G}} : \mathsf{LaxFun}(\mathcal{G}, \mathsf{Gp}) \longrightarrow \mathsf{Fib}_{\mathrm{b.o.}}(\mathcal{G}),$$

where $\mathsf{Fib}_{\mathrm{b.o.}}(\mathcal{G})$ is the full subcategory of $\mathsf{Fib}(\mathcal{G})$ determined by those fibrations which are bijective on objects.

3   Schreier invariants of groupoid extensions

In this section we see coming together the two threads running parallel in the previous one, namely that groupoid extensions are fibrations and that cocycles / cocycle conditions and cohomologous cocycles are respectively lax 2-functors / coherence conditions and lax 2-natural transformations. It is clear that the tying knot is the twisted product construction.

Recall from Example 2.19 that for any groupoid $\mathcal{G}$ the twisted product construction establishes an equivalence of categories

$$\int_{\mathcal{G}} : \mathsf{LaxFun}(\mathcal{G}, \mathsf{Gp}) \longrightarrow \mathsf{Fib}_{\mathrm{b.o.}}(\mathcal{G}).$$

Since we want to fix the fibers of our fibrations, we must fix a $\mathcal{G}_{\mathrm{obj}}$-indexed family of groups, $\mathcal{K} = \{K_A\}_{A \in \mathcal{G}}$. This family immediately determines a subcategory of $\mathsf{LaxFun}(\mathcal{G}, \mathsf{Gp})$ we already encountered, namely, the category $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$ of weak actions of $\mathcal{G}$ on $\mathsf{AUT}(\mathcal{K})$. Obviously, $\widetilde{\mathsf{Act}}(\mathcal{G}, \mathcal{K})$ is equivalent to its image in $\mathsf{Fib}_{\mathrm{b.o.}}(\mathcal{G})$ by the above equivalence. We arrive in this way at the concept of the *category of extensions of $\mathcal{G}$ by $\mathcal{K}$*, denoted $\mathrm{Ext}(\mathcal{G}, \mathcal{K})$. The problem of characterizing this category of extensions has the following answer:

Definition 3.1
An extension of a groupoid $\mathcal{G}$ by a $\mathcal{G}_{obj}$-indexed family of groups $\mathcal{K}$, is a fibration bijective on objects $P : \mathcal{E} \to \mathcal{G}$ such that for any object $A \in \mathcal{G}$ the fiber of P at A is the group K_A. If we consider $\mathcal{K}$ as a totally disconnected groupoid, an extension of $\mathcal{G}$ by $\mathcal{K}$ is just a short exact sequence of groupoids

$$1 \to \mathcal{K} \to \mathcal{E} \xrightarrow{P} \mathcal{G} \to 1,$$

that is, $\mathcal{K}$ is the kernel of P and $\mathcal{G}$ is the quotient groupoid of $\mathcal{E}$ by the normal subgroupoid $\mathcal{K}$. The category $\mathrm{Ext}(\mathcal{G}, \mathcal{K})$ is the subcategory of $\mathsf{Fib}_{\mathrm{b.o.}}(\mathcal{G})$ with objects the extensions of $\mathcal{G}$ by $\mathcal{K}$ and morphisms those morphisms of fibrations

above $\mathcal{G}$ which induce the identity on fibers. A morphism of $\text{Ext}(\mathcal{G}, \mathcal{K})$ is then a diagram of short exact sequences of groupoids

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \\
& & \| & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{G} & \longrightarrow & 1 \, .
\end{array}
$$

Let us note that any morphism of extensions is an isomorphism and therefore $\text{Ext}(\mathcal{G}, \mathcal{K})$ is a groupoid.

Using the twisted product construction it is easy to see that any weak action gives rise to an extension. Conversely, by taking fibers, any extension gives rise to a weak action, and we have:

Theorem 3.2
For any groupoid $\mathcal{G}$ and any $\mathcal{G}_{obj}$-indexed family of groups $\mathcal{K}$, there is an equivalence of categories

$$
\int_{\mathcal{G}} : \widetilde{\text{Act}}(\mathcal{G}, \mathcal{K}) \longrightarrow \text{Ext}(\mathcal{G}, \mathcal{K})
$$

which is the restriction to $\widetilde{\text{Act}}(\mathcal{G}, \mathcal{K})$ of the twisted product equivalence $\int_{\mathcal{G}}$: $\mathsf{LaxFun}(\mathcal{G}, \mathsf{Gp}) \to \mathsf{Fib}_{\text{b.o.}}(\mathcal{G})$.

According to this theorem, and with the classification problem of groupoid extensions in mind, it makes sense to define groupoid cohomology as:

Definition 3.3
Given a groupoid $\mathcal{G}$ and a family of groups $\mathcal{K}$, indexed by the objects of $\mathcal{G}$, we define the 2-dimensional cohomology of $\mathcal{G}$ with coefficients in the 2-groupoid $\mathsf{AUT}(\mathcal{K})$ as the set of connected components of the category of weak actions of $\mathcal{G}$ on $\mathcal{K}$

$$
H^2_{|\mathsf{AUT}(\mathcal{K})|}(\mathcal{G}, \mathcal{K}) = H^1\big(\mathcal{G}, \mathsf{AUT}(\mathcal{K})\big) := \widetilde{\text{Act}}[\mathcal{G}, \mathcal{K}],
$$

We use square brackets to denote connected components. It is clear that this cohomology is not functorial on $\mathcal{K}$, it is functorial in the second variable at the level of morphisms of 2-groupoids.

A weak action of $\mathcal{G}$ on $\mathcal{K}$ will be also called a *2-cocycle of $\mathcal{G}$ with coefficients in* $\mathsf{AUT}(\mathcal{K})$. Then, Proposition 2.6 can be seen as a definition (or parameterization) of 2-cocycles. Two 2-cocycles will be called cohomologous if there is a morphism between them. Example 2.11 shows that two 2-cocycles F_1 and F_2 are cohomologous if and only if there is $\boldsymbol{\tau} \in C^1(\mathcal{G}, \mathcal{K})$ such that $F_2 = \boldsymbol{\tau} \nabla F_1$.

On the other hand, Proposition 2.6 and Example 2.10 prove that for groups G and K the non abelian cohomology $H^2_{\mathsf{Aut}(K)}(G, K)$ defined above coincides with Dedecker *épaisse* 2-cohomology of G with coefficients on the $\mathsf{Aut}(K)$-crossed module K.

After this definition, Theorem 3.2 has the following immediate corollary:

Corollary 3.4 (Interpretation of groupoid cohomology)
For any groupoid $\mathcal{G}$ and any $\mathcal{G}_{obj}$-indexed family of groups $\mathcal{K}$, there is a bijection

$$
H^2_{|\mathsf{AUT}(\mathcal{K})|}(\mathcal{G}, \mathcal{K}) \cong \text{Ext}[\mathcal{G}, \mathcal{K}]
$$

between the 2-cocycles of $\mathcal{G}$ with coefficients in the 2-groupoid (crossed module) $\mathsf{AUT}(\mathcal{K})$ and the connected components of the category $\text{Ext}(\mathcal{G}, \mathcal{K})$.

4 Representation of groupoid cohomology

The objective of this section is to give a representation theorem for the 2-dimensional non abelian cohomology of groupoids defined in terms of homotopy classes of simplicial maps. This theorem can be used, for example, to classify homotopy classes continuous maps from a 1-type to a 2-type in terms of this cohomology.

The key to the representation theorem (below, Theorem 4.2) will be the construction of nerves. We will use the *geometric nerve* of a 2-category given in [7]. Let us briefly recall its definition.

We denote Δ the simplicial category, whose objects are the finite non-empty linear orders, $\mathbf{1} = [0] = \{0\}, \mathbf{2} = [1] = \{0 \leq 1\}, \ldots$ and whose arrows are functors or *monotonic* maps. If $\mathcal{A}$ is a category and there is a functor $\mathbf{i} : \Delta \to \mathcal{A}$ so that Δ can be regarded as embedded into $\mathcal{A}$, one can define the n-simplices of an object A of $\mathcal{A}$ as the arrows from $\mathbf{i}([n])$ to A, that is, the n-simplices of A are the $\mathbf{i}([n])$-elements of A,

$$\mathsf{Ner}(A)_n = \mathcal{A}\big(\mathbf{i}([n]), A\big),$$

then the functoriality of $\mathbf{i}$ provides face and degeneracy operators satisfying the simplicial identities so that $\mathsf{Ner}(A)$ becomes a simplicial set. In this way one obtains a functor $\mathsf{Ner} : \mathcal{A} \to \mathsf{Set}^{\Delta^{\mathrm{op}}} = \mathsf{Sset}$ defined on objects as

$$\mathsf{Ner}(A) = \mathcal{A}\big(\mathbf{i}(-), A\big)$$

and on arrows $f : A \to B$ in $\mathcal{A}$ via composition: $\mathsf{Ner}(f) = \bar{f}$ is the simplicial map

$$\mathsf{Ner}(A)_n \xrightarrow{\bar{f}_n} \mathsf{Ner}(B)_n, \qquad \bar{f}_n(\alpha) = f \circ \alpha.$$

Definition 4.1
If we regard Δ as a full subcategory of Cat, since Cat is, in turn, a full subcategory of $2\text{-}\mathsf{Cat}_{\mathrm{lax}}$, we have a full embedding $\mathbf{i} : \Delta \to 2\text{-}\mathsf{Cat}_{\mathrm{lax}}$ and, by the above process, a functor

$$\mathsf{Ner} : 2\text{-}\mathsf{Cat}_{\mathrm{lax}} \to \mathsf{Sset}$$

which we take as the definition of the geometric nerve of 2-categories.

Then the nerve of a 2-category $\mathbf{A}$ has:

- The objects of $\mathbf{A}$ as 0-simplices,
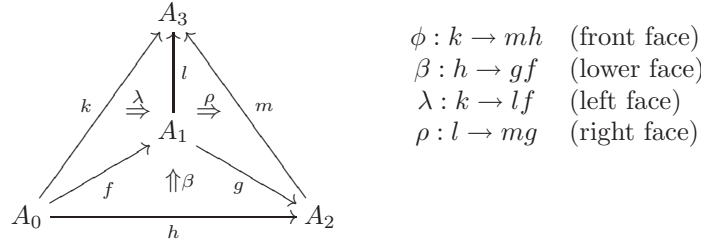- the arrows $A_0 \xrightarrow{f} A_1$ of $\mathbf{A}$ as 1-simplices, with faces

$$d_0(f) = A_1 \quad \text{and} \quad d_1(f) = A_0,$$

- the diagrams $\Delta = (g, h, f; \alpha)$ of the form

(where $\alpha : h \to gf$ is a 2-cell in $\mathbf{A}$) as 2-simplices, whose faces are the 1-simplices opposite to the indicated vertex (so, in the above example, $d_0(\Delta) = g$, etc.).

- the "commutative" tetrahedral Θ of the form

$$
\begin{array}{ll}
\phi : k \to mh & \text{(front face)} \\
\beta : h \to gf & \text{(lower face)} \\
\lambda : k \to lf & \text{(left face)} \\
\rho : l \to mg & \text{(right face)}
\end{array}
$$

as 3-simplices. The face operators for such tetrahedron are, as in the case of a 2-simplex, the 2-simplices opposite to the vertex indicated by the operator $\big($so, for example, $d_3(\Theta) = (m, k, h; \phi)\big)$.

For dimensions higher than 2, $\mathsf{Ner}(\mathbf{A})$ is coskeletal. We also have that the geometric nerve functor $\mathsf{Ner} : 2\text{-}\mathsf{Cat}_{\mathrm{lax}} \to \mathsf{Sset}$ is full and faithful (see [7, Proposition 3.3]) and that a lax natural transformation between two lax functors $F, G : \mathbf{A} \to \mathbf{B}$ induces a homotopy between the simplicial maps $\mathsf{Ner}(F)$ and $\mathsf{Ner}(G)$. Furthermore, if $\mathbf{B}$ is a 2-groupoid, a homotopy $\mathsf{Ner}(F) \to \mathsf{Ner}(G)$ exists if and only if there is a lax 2-natural transformation $\alpha : G \to F$ (see [7, Proposition 3.5]).

If we particularize the above results to weak actions of a groupoid $\mathcal{G}$ on a family of groups $\mathcal{K}$, we have that weak actions of $\mathcal{G}$ on $\mathcal{K}$ are in bijective correspondence with simplicial maps from the nerve of $\mathcal{G}$ to the nerve of $\mathsf{AUT}(\mathcal{K})$. Moreover a morphism $\boldsymbol{\alpha} : F_1 \to F_2$ between two weak actions induces a homotopy $\mathbf{h} = \mathsf{Ner}(\boldsymbol{\alpha}) : \mathsf{Ner}(F) \to \mathsf{Ner}(G)$ with the property that $h_{00} : \mathsf{Obj}(\mathcal{G}) \to \mathsf{Arr}(|\mathsf{AUT}(\mathcal{K})|)$ takes any object of $\mathcal{G}$ to an identity in $|\mathsf{AUT}(\mathcal{K})|$. We call a *normalized homotopy* to any homotopy satisfying this property, and we have

Theorem 4.2 (Representation by simplicial maps)
Given a groupoid $\mathcal{G}$ and any family $\mathcal{K}$ of groups indexed by the objects of $\mathcal{G}$, there is a bijection

$$
H^2_{|\mathsf{AUT}(\mathcal{K})|}\big(\mathcal{G}, \mathcal{K}\big) \cong \mathsf{Sset}_*\big[\mathsf{Ner}(\mathcal{G}), \mathsf{Ner}\big(\mathsf{AUT}(\mathcal{K})\big)\big],
$$

where the star and the square brackets mean normalized homotopy classes of simplicial maps.

References

[1] F. Borceux. *Handbook of Categorical Algebra 2; Categories and Structures*, **Volume** 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge Univ Press, 1994.

[2] L. Breen. Bitorseurs et cohomologie non abelienne. In *The Grothendieck Festschrift*, **Volume** I, number 86 in Progr. Math., pages 401–476. Birkhäuser Boston, Boston, MA, 1990.

[3] L. Breen. Théorie de Schereier supérieure. *Ann. Sci. École Norm. Sup. (4)*, 25, 1992.

[4] L. Breen. On the classification of 2-gerbes and 2-stacks. *Asterisque*, (225):1–160, 1994.

[5] Ronald Brown and Timothy Porter. On the Schreier theory of non-abelian extensions: Generalisations and computations. *Proceedings of the Royal Irish Academy*, 96A(2):213–227, Dec 1996.

[6] M. Bullejos and A. M. Cegarra. A 3-dimensional non-abelian cohomology with applications to homotopy classification of continuous maps. *Canad. J. Math.*, 43 - 2:265–296, 1991.

[7] M. Bullejos, E. Faro, and V. Blanco. A full and faithful nerve for 2-categories. Available at `http://arXiv.org/abs/math.CT/0406615`.

[8] P. Dedecker. Les foncteurs Ext_Π, $\mathbf{H}^2_\Pi$ et H^2_Π non abeliens. *C. R. Acad. Sc. Paris*, 258 Groupe 1:4891–4895, 1964.

[9] A. Grothendieck. *Catégories fibrées et déscente (SGA I)*. Number 224 in Lecture Notes in Math. Springer, New York, 1971.

[10] G. Moore and N. Seiberg. Classical and quantum field theory. *Comm. Math. Phys.*, 123:177–254, 1989.

[11] O. Schreier. Uber die erweiterung von gruppen I. *Monatshefte für Mathematik und Physik*, 34:165–180, 1926.

[12] O. Schreier. Uber die erweiterung von gruppen II. *Abh. Math. Sem. Hamburg*, 4:321–346, 1926.

[13] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85:91–109, 1979.

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